Difficulties in Modeling Interest Rates

An Honors Thesis (HONORS 499)

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Abstract

This paper discusses two types of basic interest rate models: the Vasicek model and Cox-Ingersoll-Ross model. The mathematics behind interest rate modeling is extremely complex, so this paper does not review proofs of the stochastic processes behind these models. Instead, the paper focuses on various techniques to estimate the parameters of the models. It also discusses the difficulties in creating and implementing these models using Microsoft Excel.

Acknowledgements

I would like to thank Professor Gary Dean for his willingness to help me on this thesis. He provided excellent guidance and insight on this project. Without his help, this paper would not have been possible.

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Introduction

Any investor knows that financial markets are volatile. For years researchers have studied the financial markets looking for trends in data that may help investors predict the future of these markets. Due to the extremely complex nature and plethora of variables to be considered, investors are still unable to predict the markets with much certainty; however, research hasn't been an entire waste of time. Many models have been created which are able to help investors manage their risk.

These models have become especially important in recent years as the derivatives market has exploded. The introduction of these complex financial instruments has commanded the need for better models of prices of their underlying assets. Before issuing a new financial product, a company must understand the cash flows for the product. This often relies on scenario testing of stocks, bonds, interest rates, and other assets. Small changes in interest rates can result in millions of dollars of changes in asset values. For this reason, many models incorporate different scenarios to illustrate cash flows under various potential conditions.

This paper examines the implementation of two interest rate models: the Vasicek model, created in 1977, and the Cox-Ingersoll-Ross model from 1985. These models are classified as "short rate models" opposed to "no-arbitrage models." These short rate models fit the parameters of the model to historical data, which may allow for arbitrage. The term structure of interest rates is then determined through the simulation of the model using the estimated parameters. While this paper does not examine the term structure of interest rates, this characteristic distinguishes the Vasicek and Cox-Ingersoll-Ross short rate models from no arbitrage models and is important to point out. The no arbitrage models, in contrast, use the current term structure of interest rates, as implied by zero coupon bonds of various maturities, to develop a model that does not allow for arbitrage. Both types of models have their advantages and disadvantages which will be examined later in the paper through the simulation of the Vasicek and Cox-Ingersoll-Ross models.

These models are also classified as one-factor models. This means that the predicted interest rate is a function of the previous interest rate. Although, one-factor models are elementary among the new multi-factor models, they provide a good introduction to the study of interest rates. The assumptions and details of these models are incorporated into other models; therefore, a study of the both the accomplishments and shortcomings of these models provides a strong background for understanding the dynamics of interest rates.

Vasicek Model

One of the most basic interest rate models was originally introduced by Vasicek in 1977. The interest rates modeled in this paper are based off of the daily interest rate reported by the 13-week Treasury Bill as stated on Yahoo! Finance. Due to extra risk factors that are not considered in the Vasicek model, only "risk-free" bonds were modeled. (Treasury bills, notes, and bonds issued by the United States government are assumed in most financial studies to be risk-free.) The general equation for this stochastic process is shown below:

$$dr = u(r)dt + \sigma(r)dW$$

where:
dr = change in interest rate
$u(r)$ = drift rate
dt = change in time

$\sigma(r)$ = standard deviation of the interest rate
dW = random variable distributed normally with mean 0 and variance σ^2

This can be specialized for the Vasicek model. The equation as used in the modeling of interest rates using the Vasicek model is shown below:

$$r_{k+1} = r_k + a(b - r_k)dt + \sigma Z$$

where:

r_k = interest rate in period k
a = rate of mean reversion
b = mean
σ = standard deviation
Z = random variable distributed normally with mean 0 and variance σ^2
dt = change in time

For r_0, the interest rate used is the historic interest rate for the beginning day of the time period modeled as found on Yahoo! Finance. For periods 1 through n, the interest rate used is the interest rate from period k-1 as generated by the equation. The change in time can be assumed to be 1 throughout the Vasicek models used in this paper. The random variable *Z* is particularly important in the development of the model. It accounts for Vasicek's idea that interest rates follow a stochastic process known as the Ornstein-Uhlenbeck process. While the details of this process are beyond the scope of this paper, some of the assumptions of this process are important underlying assumptions of this model.

The first assumption states that changes in interest rates are continuous and therefore interest rates will pass through every intermediate value before reaching a new value. Obviously, in practice this is not true; interest rates can and do jump around. In the modeling observed in this paper, this assumption does not seem to strongly hinder the model. The second assumption states that the random term, Z in the model, is distributed normally with mean zero and variance σ^2. This assumption of a normal distribution allows calculations to be performed much easier. Finally, the last assumption states that interest rates follow a process known as mean reversion. Mean reversion requires that if an interest rate becomes too high in relation to its historic mean, it will fall, and conversely if an interest rate becomes too low in relation to its historic mean, it will rise. This reversion is shown in the term $b - r_k$, and the rate of reversion is measured by the parameter a. The larger a is, the faster the interest rate reverts back to its mean. It must be reiterated that these are not all the assumptions of the Ornstein-Uhlenbeck process, but they are the assumptions that are easily described and most visible in the Vasicek model.

Four measures for the estimation of parameters a, b, and σ were taken. First, the standard deviation of the historic interest rates for the time period being modeled was used as a measure of σ and similarly, the mean of the historic interest rates was used as a measure of b. The rate at which the interest reverted back to the mean, or a, was found by taking the average of the growth of the absolute value of $b - r_k$. This model produced highly volatile results as illustrated in the graph of simulated interest rates versus historic daily interest rates for the one year time period of the 13-week Treasury Bill shown below:

Figure 1: Daily Historic and Simulated Interest Rates for 13-Week Treasury Bill (Version One)

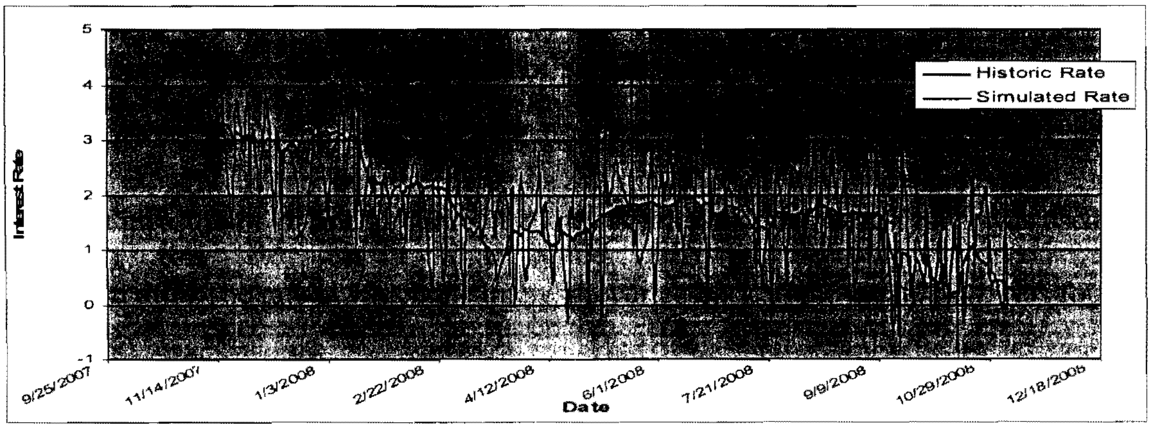


Table 1: Results from Simulated Interest Rates for 13-Week Treasury Bill (Version One)

	Parameters for r_k	*Statistical Results for historic interest rate*		*Statistical Results for r_k*	
a	0.3589	**mean**	1.7252	**mean**	1.7651
b	1.7252				
σ	0.7716	**standard deviation**	0.7716	**standard deviation**	0.9646

It's important to note that the graph of the simulated rate is just one example of many possible outcomes. Because the Excel model uses a random number generator as part of its input, there are an infinite number of possibilities. Though the actual interest rate values produced will vary with each simulation, the mean reversion trend will not. The graph above clearly shows that the model over/under predicts the interest rate each time and then over/under corrects itself. This is due to a high estimation for the standard deviation term σ . To fix this problem, a linear regression was performed on the historic data. If the Vasicek equation is rewritten in the form:

$$y_i = mx_i + h + u$$

where:

$y_i = r_{k+1} - r_k$

$x_i = r_k$

$m = a$

$h = ab$

$u = \sigma Z$

a simple linear regression can be performed using only the historic interest rate from the observed time period. Parameters a and b can be solved for using the coefficient and intercept from the regression results, and the standard error of the y-value can be used as an estimate for σ. Using these parameters in the model for the 13-week Treasury Bill from the same time period as previously simulated, gives the graph below:

Figure 2: Daily Historic and Simulated Interest Rates for 13-Week Treasury Bill (Version Two)

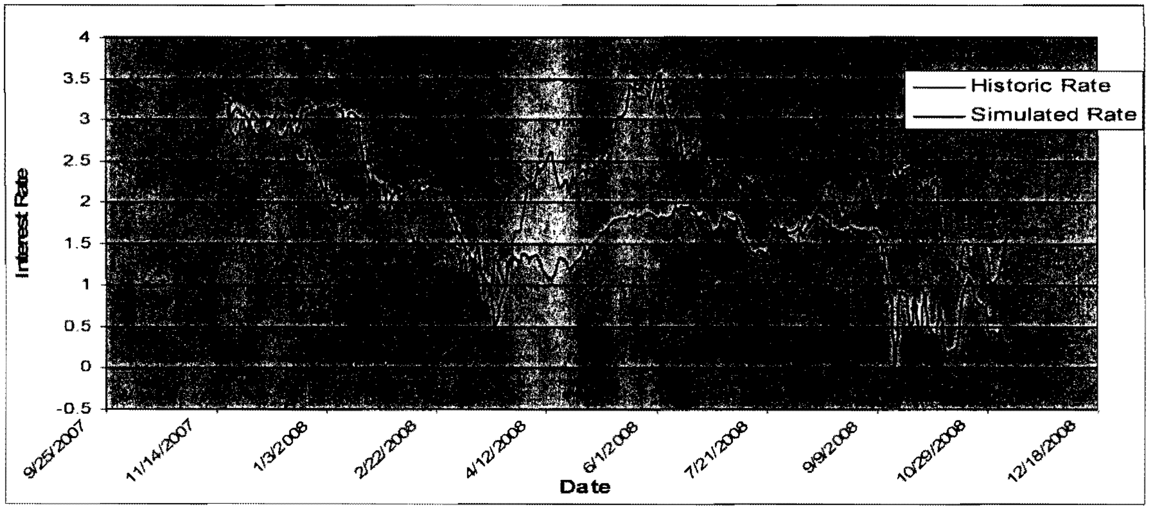


Table 2: Results from Simulated Interest Rates for 13-Week Treasury Bill (Version Two)

Parameters for r_k		*Statistical Results for historic interest rate*		*Statistical Results for r_k*	
a	0.0187	**mean**	1.7252	**mean**	2.1271
b	1.0787				
σ	0.0187	**standard deviation**	0.7716	**standard deviation**	0.6159

This graph appears to be a much more accurate representation of a possible random outcome of the interest rate. The simulated interest rate has similar volatility characteristics and also mimics some of the high and low trends in the graph.

Although this simulation does look like a possible outcome for interest rates over the given time period, the Vasicek model still has a few hindrances to becoming a more accurate interest rate model. First, as shown in figure one, interest rates can go negative in the Vasicek model. This is due to the assumption that interest rates are distributed normally, which allows for a positive probability of a negative interest rate. A negative interest rate will be more likely as standard deviation increases since the probability of generating an interest rate value much less than zero is greater. While many financial researchers believe that this is a downfall of the Vasicek model since negative interest rates rarely occur in practice, Moorad Choudhry argues in *Analysing and Interpreting the*

Yield Curve that this idea is not a weakness of the model because a negative interest rate is not entirely unrealistic and it will only appear in the model, and real life, under extreme conditions.

Another possible flaw of the Vasicek model is the assumption of a constant variance. The graph of historical interest rates shows that the volatility of interest increases and decreases throughout time. This is further illustrated by taking the standard deviation over the thirty year time period (that is examined in the model simulations) in thirty day increments. Each increment in figure 3 is graphed according to the first date in the thirty day increment. Figure 3 shows exactly how much the standard deviation changes through time.

Figure 3: Standard Deviation of Simulated Interest Rates over Time

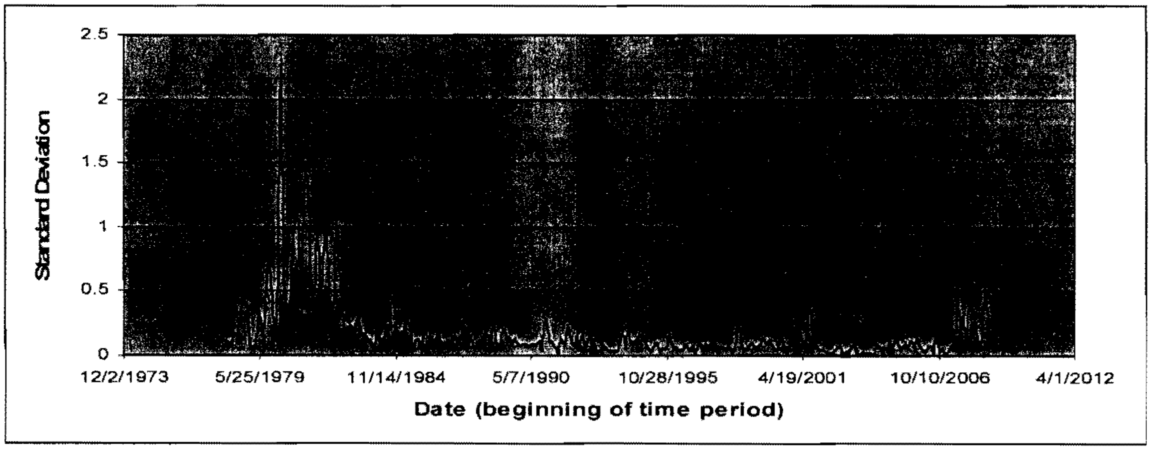


Figure 4: Daily Changes in Interest Rates

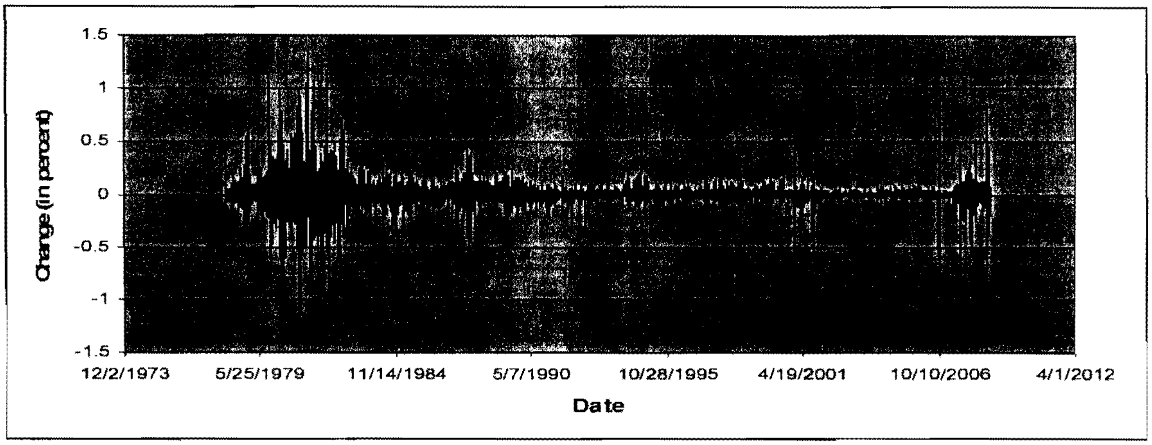


Figure four shows the daily change in interest rates and represents the type of graph that financial professionals use to look at volatility over time. The greater changes obviously indicate a greater volatility during that time. As the time period modeled increases, the volatility parameter in the model tends to increase as well due to more extreme changes being included in the sampling time period. The table below looks at a sample of thirty simulations of the three month, one year, five year, ten year, twenty year, and thirty year time periods that were modeled in Excel.

Table 3: Difference in Simulated Standard Deviation and Observed Standard Deviation

Time	Starting Date	Observed Standard Deviation	Simulated Standard Deviation	Difference from Observed Standard Deviation
3 months	8/11/2008	0.5807	0.4821	0.0986
1 year	11/16/2007	0.7716	0.7899	0.0183
5 years	1/2/2003	1.5509	1.0246	0.5263
10 years	1/2/1998	1.6831	1.5506	0.1325
20 years	1/4/1988	1.9551	1.8029	0.1522
30 years	1/4/1978	3.1801	2.7528	0.4273

The simulated interest rates tend to have a smaller standard deviation than the historic standard deviation. However, these differences are small even for the thirty year time period. Therefore, despite the assumption of a constant variance, the simulated interest rates are producing graphs of interest rates that can mimic the historical graph of interest rates in a given time period.

Also varying through time are the macroeconomic factors in interest rates. Interest rates succumb to the invisible hand of the market and many other economic factors that are difficult to factor into the Vasicek model. Most of these factors are considered random and can be accounted for in the random variable of the model. Occasionally, however, the Fed has stepped in and changed interest rates. These changes are not random and therefore cannot be incorporated into the model, yet the changes still impact the value of the interest rate. This demonstrates one of the problems with one-factor models.

Finally, the Vasicek model ignores an assumption that is important for the pricing of some derivatives: arbitrage. The Vasicek model is considered a short rate or equilibrium model, not an arbitrage-free model. According to Choudhry, an equilibrium model is one that "is derived from (or consistent with) a general equilibrium model of the economy." Therefore, these models are useful in markets that do not have much historical data such as an emerging market.

Cox-Ingersoll-Ross Model

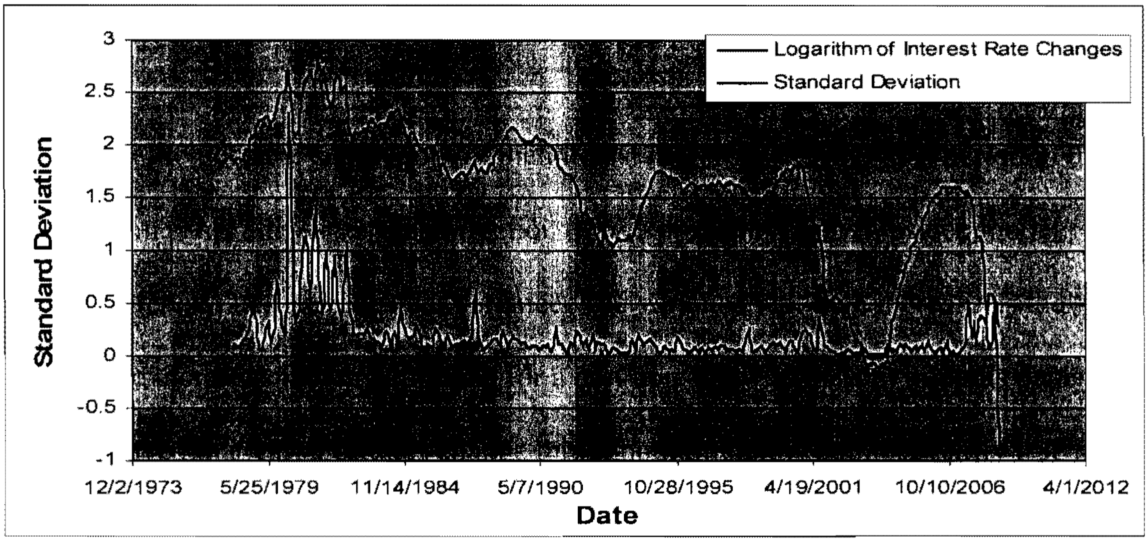
The Cox-Ingersoll-Ross model builds off of the Vasicek model and makes some changes to account for the downfalls of the Vasicek model. Because of the similarity to the Vasicek model, it also incorporates some of the same underlying assumptions such as that the random variable follows a stochastic process known as the Ornstein-Uhlenbeck process. Also, as with the Vasicek model, the daily rates of the 13-week Treasury Bill as recorded by Yahoo! Finance were used in the modeling. The general equation used in the Excel models is as follows:

$$r_{k+1} = r_k + a(b - r_k)dt + \sigma\sqrt{r_k}Z$$

Where:
r_k = interest rate in period k
a = rate of mean reversion
b = mean
σ = standard deviation
Z = random variable distributed Normal with mean 0 and variance σ^2
dt = change in time

This model includes the square root of the interest rate from the previous period in the random term. This allows for the correction of two problems of the Vasicek model: negative interest rates and constant variance. The variance will then increase as the interest rate increases thereby allowing for a non-constant variance. While this paper already proved that variance does change throughout time, it is not necessarily true that there is a direct relationship between interest rate and variance. Figure five below examines the relationship between interest rates changes and standard deviation:

Figure 5: Logarithm of Interest Rates against Standard Deviation over Time



The graph shows that the standard deviation does not necessarily increase as interest rates increase or decrease as the interest rates decrease, which is one downfall of this assumption. Including the square root of the interest rate in the error term, however, does allow for the interest rate to remain positive. Robert McDonald in *Derivative Markets* states that if the interest rate is zero, then the drift in the rate will be positive and the error term will be zero, so the predicted interest rate will become positive. Early drafts of the model did not demonstrate this feature, however.

Figure 6: Observed and Simulated Interest Rates using Cox-Ingersoll-Ross Model (Version One)

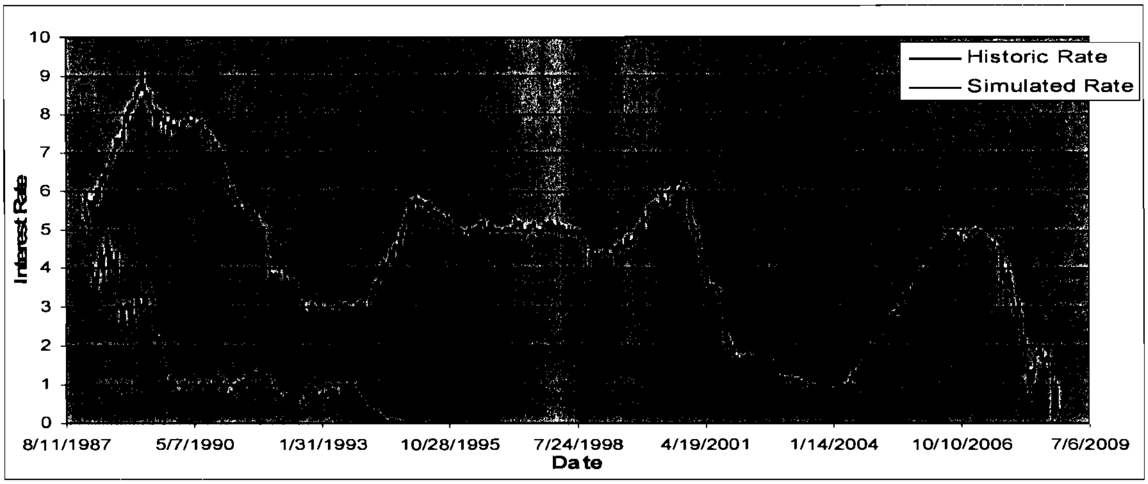


Table 4: Results from Simulated Interest Rates for 13-Week Treasury Bill (Version One)

Parameters for r_k		*Statistical Results for historic interest rate*		*Statistical Results for r_k*	
a	0.0001	**mean**	4.3082	**mean**	1.4398
b	-3.5660				
		standard deviation	1.9551	**standard deviation**	1.2536
σ	0.0605				

In the figure above, the parameters were estimated by a linear regression (the same procedure as used in the second version of the Vasicek model). The parameters a and b were therefore found by the coefficients of the regression and the standard error of the y-value was used as an estimate for σ. This linear regression technique produced inaccurate value for the parameters since the simulated interest rates went negative, which isn't allowed in the assumptions of the model. (If a negative interest rate is predicted, the next term would have to take the square root of a negative interest rate, which is clearly impossible to do in terms of real numbers.) Table four shows the parameter values used in the model in figure six. With a negative mean, or b, value the drift term will always be negative and the error term is too small, due to a low standard deviation, to keep the interest rate positive for a thirty year time span. To fix this problem, a different regression equation was used:

$$y_i = mx_i + u$$

Where:

$$y_i = \frac{r_{k+1} - r_k}{\sqrt{r_k}}$$

$$x_i = \frac{b - r_k}{\sqrt{r_k}}$$

b = historic mean

$m = a$

$u = \sigma Z$

For shorter time periods interest rates do not tend to go negative, but it is more likely in longer time periods as shown below in the ten-year sampling:

Figure 7: Observed and Simulated Interest Rates using Cox-Ingersoll-Ross Model (Version Two)

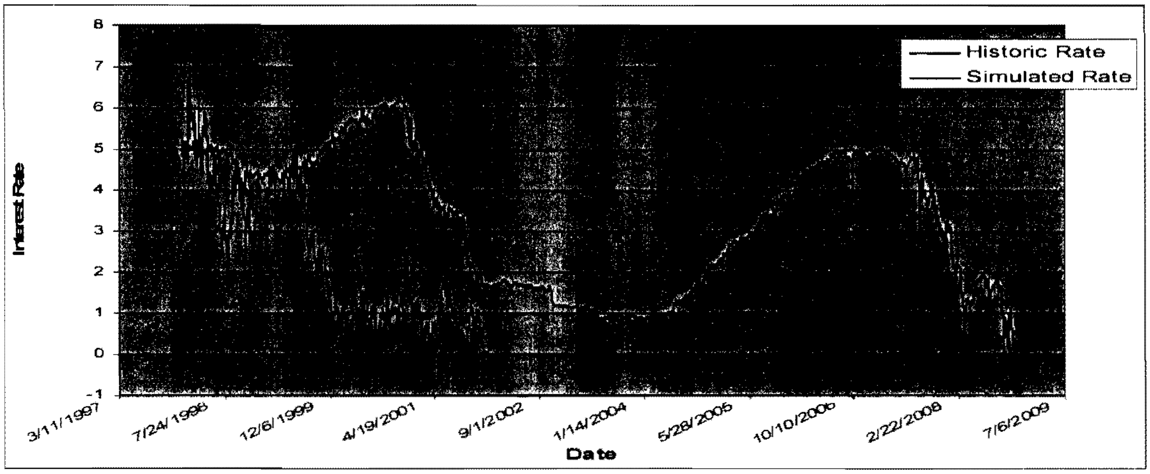


Table 5: Results from Simulated Interest Rates for 13-Week Treasury Bill (Version Two)

Parameters for r_k		*Statistical Results for historic interest rate*		*Statistical Results for r_k*	
a	0.0007	**mean**	3.3166	**mean**	4.0727
b	3.3166				
σ	0.1223	**standard deviation**	1.6831	**standard deviation**	0.9486

This again indicates that the parameters were not chosen correctly. While the parameter b was not negative this time, it is still smaller than the initial interest rate, or r_0 value. This causes the drift term to be negative for the first couple years of simulation. With a small standard deviation value again, the error term is too small to pull the interest rate up over time. To fix the problem this time, the regression technique was changed. In any linear regression, the linear coefficients are found by minimizing the equation:

$$\sum_{i=1}^{n}(y_i - h - mx_i)^2$$

The regression assumes that $y_i - h - mx_i$ is distributed normally with mean zero and variance σ^2. The Cox-Ingersoll-Ross model assumes that this error term is actually distributed normally with mean zero and variance $\sigma^2\sqrt{x_i}$. The probability density function is changed accordingly and then the maximum likelihood function is found (a proof of this can be found in appendix A). Taking partial derivatives, setting the partial derivatives equal to zero and solving for the coefficients leads to the following equations for *a* and *b*:

$$a = -m \qquad b = \frac{h}{a}$$

This computation provided slightly better results. The simulated rate did not go negative, but it did prove to be much more volatile than the historic rate. The thirty year time period illustrated in figure eight best illustrates this attribute:

Figure 8: Observed and Simulated Interest Rates using Cox-Ingersoll-Ross Model (Version Three)

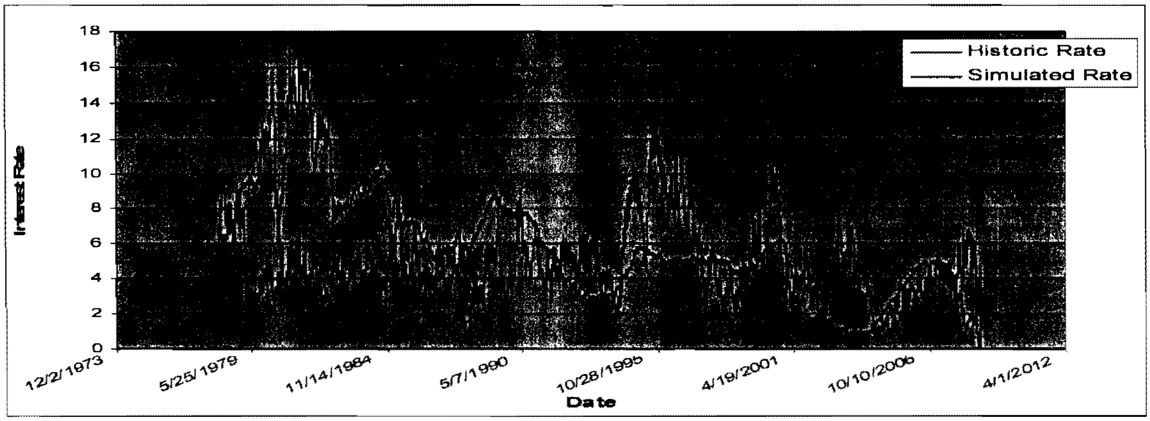


Table 6: Results from Simulated Interest Rates for 13-Week Treasury Bill (Version Three)

Parameters for r_k		Statistical Results for historic interest rates		Statistical Results for r_k	
a	0.0035	**mean**	5.8437	**mean**	4.7987
b	5.6263				
σ	0.1115	**standard deviation**	3.1801	**standard deviation**	2.0551

Though the standard deviation is low, the *a* value is very high. This makes the simulated rate revert back to the mean too quickly, often over/under predicting the historic rate.

Finally, the last method taken to find parameters for the Cox-Ingersoll-Ross model was found in the paper *Parameter Estimation and Bias Correction for Diffusion Processes*. A continuous time diffusion process can be used to model interest rates and from this the following calculations for the parameters were found:

$$a = -\frac{1}{\delta}\ln(\beta_1) \text{ with } \delta = 1 \qquad b = \beta_2 \qquad \sigma^2 = \frac{2a\beta_3}{1-\beta_1^{\ 2}}$$

where:

$$\beta_1 = \frac{\frac{1}{n^2}\sum_{i=1}^{n} x_i \sum_{i=1}^{n}\frac{1}{x_{i-1}} - \frac{1}{n}\sum_{i=1}^{n}\frac{x_i}{x_{i-1}}}{\frac{1}{n^2}\sum_{i=1}^{n} x_{i-1}\sum_{i=1}^{n}\frac{1}{x_{i-1}} - 1}$$

$$\beta_2 = \frac{\frac{1}{n}\sum_{i=1}^{n} x_i x_{i-1} - \beta_1}{(1-\beta_1)\frac{1}{n}\sum_{i=1}^{n} 1/x_{i-1}}$$

$$\beta_3 = \frac{1}{n}\sum_{i=1}^{n}[x_i - x_{i-1}\beta_1 - \beta_2(1-\beta_1)]^2\left(\frac{1}{x_{i-1}}\right)$$

These parameters provided non-negative, but still volatile results. Figure nine shows this below:

Figure 9: Observed and Simulated Interest Rates using Cox-Ingersoll-Ross Model (Version Four)

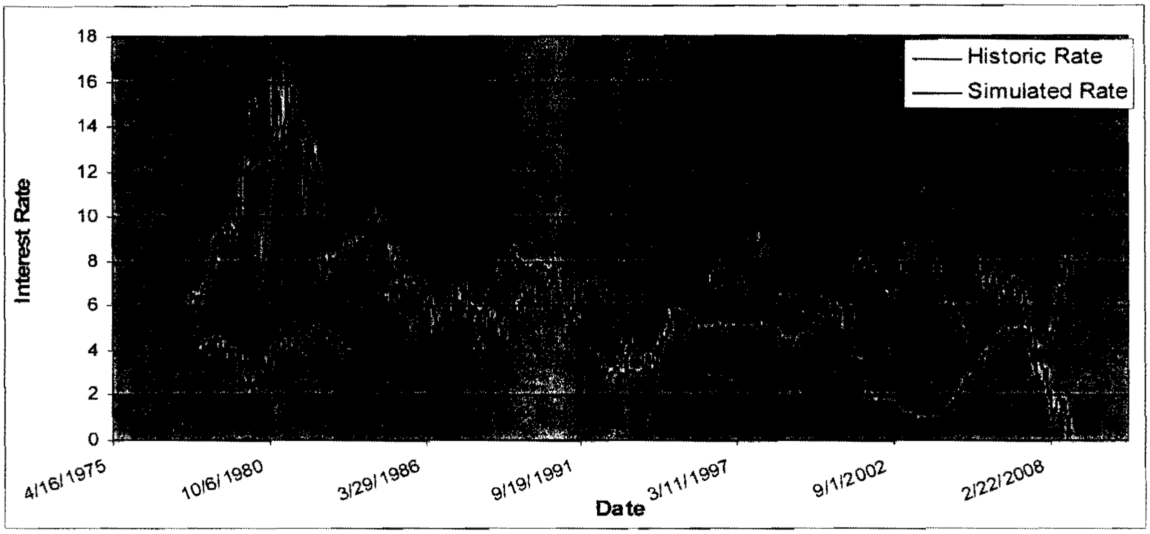


Table 7: Results from Simulated Interest Rates for 13-Week Treasury Bill (Version Four)

Parameters for r_k		*Statistical Results for historic interest rate*		*Statistical Results for r_k*	
a	0.0035	**mean**	5.8479	**mean**	5.8319
b	5.6263				
		standard deviation	3.1777	**standard deviation**	1.5512
σ	0.0575				

This model produces simulated interest rates that are very similar to version three of the Cox-Ingersoll-Ross model. In fact, a comparison of the parameters shows that the models are indeed quite alike:

Table 8: Parameter Comparison for Cox-Ingersoll-Ross model Version Three and Version Four

	3 months		**1 year**		**5 years**	
	Version 3	*Version 4*	*Version 3*	*Version 4*	*Version 3*	*Version 4*
a	0.1965	0.2264	0.0885	0.0926	0.0167	0.0140
b	0.8933	0.8788	1.5932	1.5932	2.6715	2.8782
σ	0.2236	0.4725	0.1425	0.2425	0.0803	0.1051

	10 years		**20 years**		**30 years**	
	Version 3	*Version 4*	*Version 3*	*Version 4*	*Version 3*	*Version 4*
a	0.0105	0.0106	0.0063	0.0063	0.0035	0.0035
b	3.1475	3.1475	4.1391	4.1391	5.6263	5.6263
σ	0.0701	0.0594	0.0605	0.0594	0.1115	0.0575

The 20 year and 30 year time periods parameter estimations result in identical values for a and b. The other time periods have a and b values that are not identical, but still very close to one another. The standard deviation estimate is also quite similar through most of the models, except for the three month models.

The Cox-Ingersoll-Ross model is very sensitive to the parameters chosen, but if the correct parameters are chosen then it can be a useful model. Some researchers would say that the Cox-Ingersoll-Ross model is an improvement from the Vasicek model since it does not allow for negative interest rates, which rarely occur in real life, and it allows for a changing variance which occurs in practice. Also, once the right technique is found for parameter estimation, it is not much more difficult to implement than the Vasicek model.

Still, figures eight and nine show that finding the right value for the standard deviation parameter for the Cox-Ingersoll-Ross model is difficult. Although the Cox-Ingersoll-Ross model accounts for a non-constant variance, unlike the Vasicek model, the method for incorporating a non-constant variance into the model is not strong enough to predict interest rates in long time periods with much accuracy.

There are situations where the Cox-Ingersoll-Ross and Vasicek model can better simulate interest rates similar to the historical rates. As previously discussed, shorter time periods that allow for a more constant variance provide fairly accurate simulations. Also, in the Excel models arbitrary time periods were chosen, but if a time period of low volatility is examined, such as in the year 2002, the Cox-Ingersoll-Ross (CIR) model appears to have a much better fit.

Figure 10: Simulated Interest Rates during a Time of Low Volatility using CIR Version Three

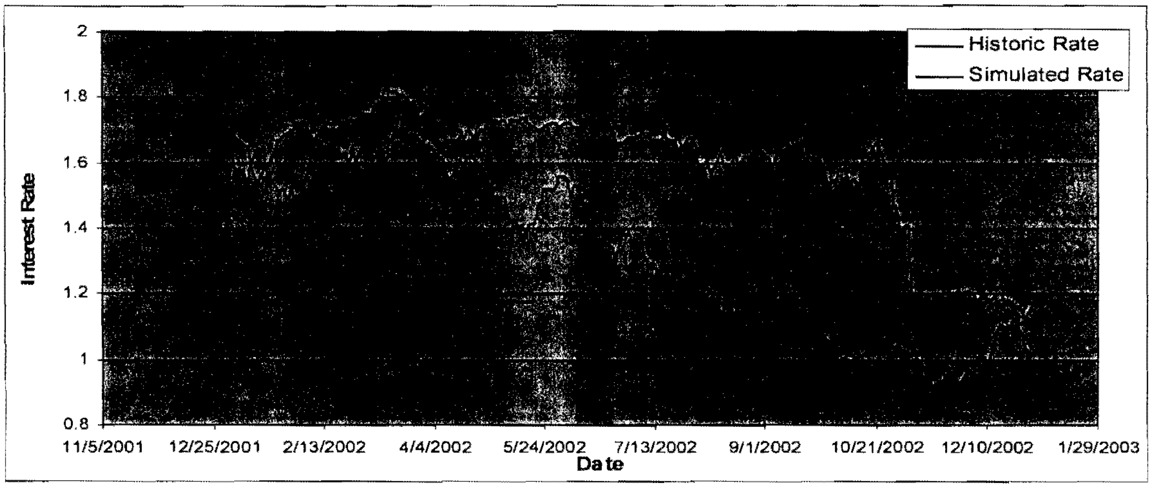


Table 9: Results from Simulated Interest Rates for Low Volatility (Version Three)

Parameters for r_k		*Statistical Results for historic rate*		*Statistical Results for r_k*	
a	-0.0016	**mean**	1.5943	**mean**	1.3287
b	2.9122				
σ	0.0223	**standard deviation**	0.1856	**standard deviation**	0.2461

Figure 11: Simulated Interest Rates during a Time of Low Volatility using CIR Version Four

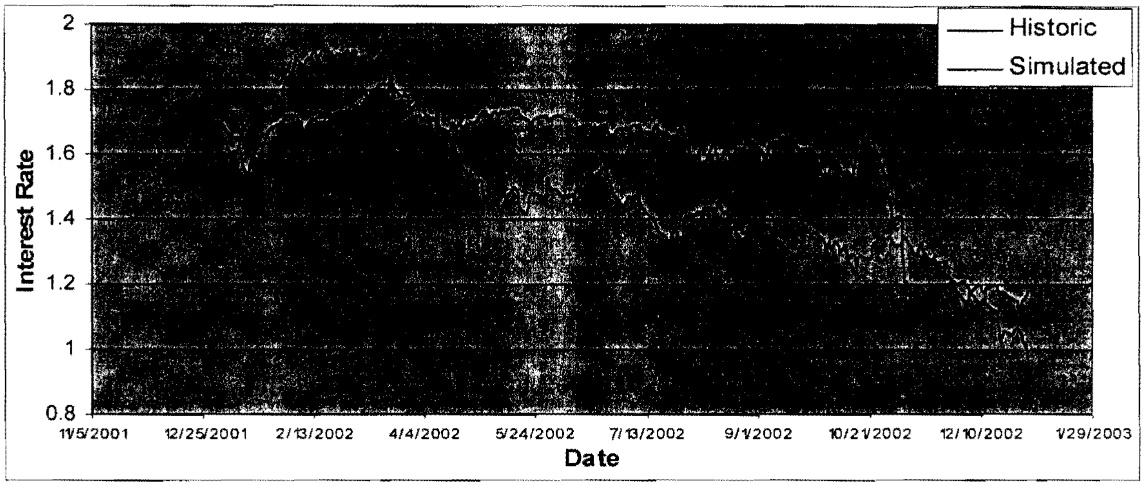


Table 10: Results from Simulated Interest Rates for Low Volatility (Version Four)

Parameters for r_k		*Statistical Results for historic rate*		*Statistical Results for r_k*	
a	-0.0016	**mean**	1.5943	**mean**	1.4815
b	2.9122				
σ	0.0179	**standard deviation**	0.1856	**standard deviation**	0.2290

Again, both version three and four give similar simulations of the interest rate, but in this period of low volatility they both produce interest rates that are very close to the historic interest rate. This shows that the Cox-Ingersoll-Ross model could be best employed in a time where low volatility is expected.

The Cox-Ingersoll-Ross model is very sensitive to the parameters chosen, but if the correct parameters are chosen then it can be a useful model. During time periods with low volatility, one can simulate interest rates accurately, while time periods with high volatility are more difficult to model. Some researchers would say that the Cox-Ingersoll-Ross model is an improvement from the Vasicek model since it does not allow for negative interest rates, which rarely occur in real life, and it allows for a changing variance which occurs in practice. Also, once the right technique is found for parameter estimation, it is not much more difficult to implement than the Vasicek model.

Conclusion

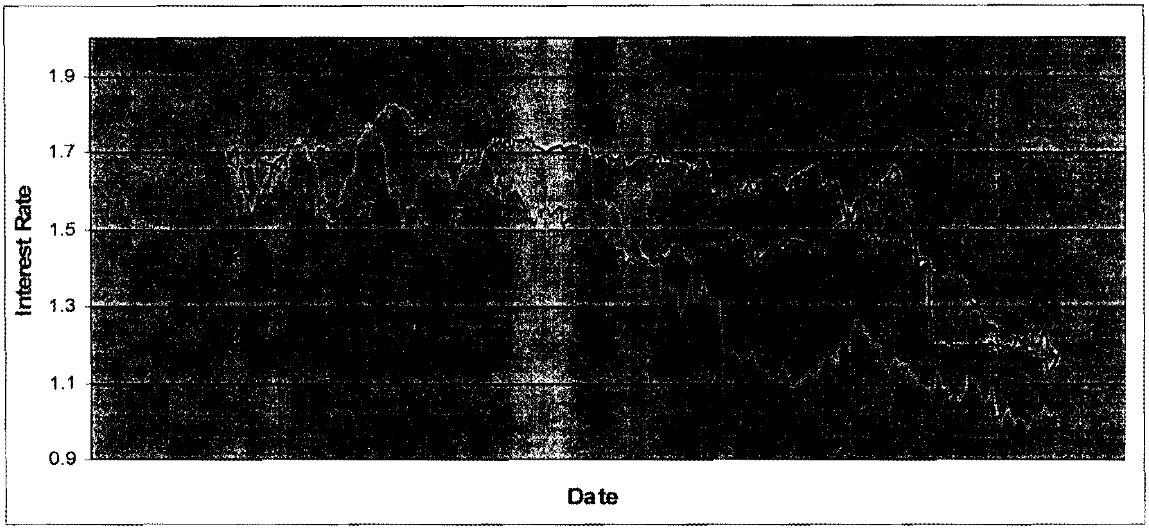
Interest rate models are an important tool for financial companies trying to manage risk and price complicated financial products. Implementing interest rate models can be difficult, however. There are a multitude of techniques to estimating the parameters; this paper illustrates the estimations that empirical studies indicate work best during an arbitrary time period. When the future volatility is known and expected to be low, the Vasicek and Cox-Ingersoll-Ross models are able to simulate even more accurately during this time period.

More specifically, the findings in this paper show that the Vasicek model accurately simulates interest rates using a simple linear regression estimate for parameters. In this case, parameters a and b are solved for using the regression coefficients and the parameter sigma is the standard error of the y-value. Figure 2 shows that this is a much better simulation of historic interest rates than figure one, which uses the historic mean and standard deviation over the time period modeled to estimate parameters b and sigma and the average of the growth of the absolute value of $b - r_k$ is used to estimate the a value.

The Cox-Ingersoll-Ross model closely resembles the Vasicek model, but attempts to improve upon the weaknesses in the Vasicek model, specifically the assumption of a constant variance over the time period and the ability to produce negative interest rates. The Cox-Ingersoll-Ross model incorporates the square root of the interest rate in period k-1 to model the interest rate in period k. This therefore allows for a non-constant variance and implies that interest rates cannot go negative. Many methods for estimating the parameters for this model were looked at, but it appears in figures 8 and 9 that the technique of maximum likelihood estimation or using a diffusion process to estimate the parameters produce interest rates that most closely resemble the historic rates.

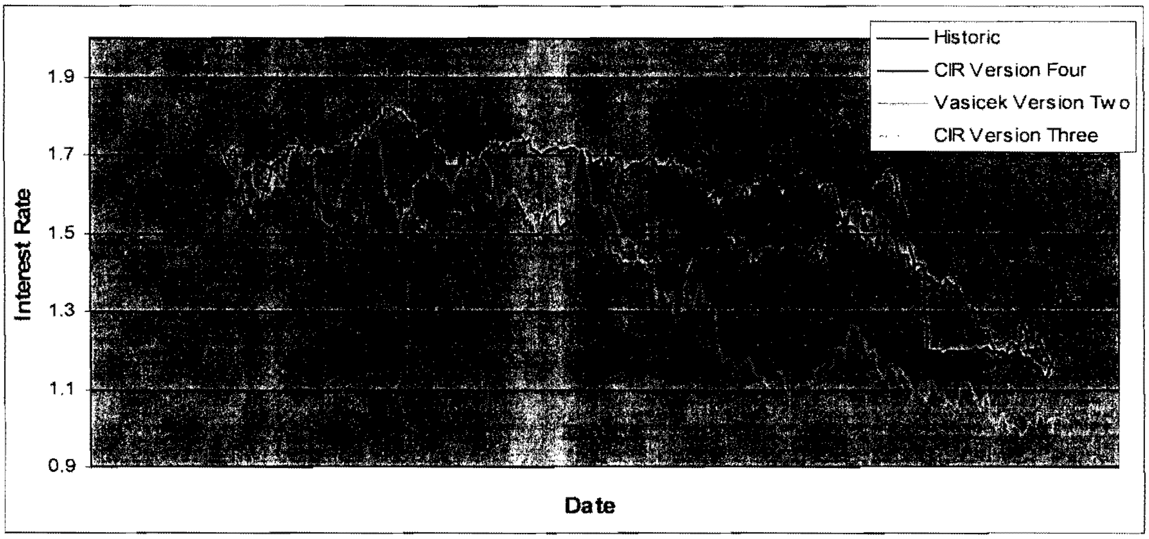
Versions three and four of the Cox-Ingersoll-Ross model and version two of the Vasicek model produce interest rates so close to the historic rates that the reader can challenge him/herself by trying to distinguish between the three models and the historic rates in the figure below:

Figure 12: Three Model Simulations and Historic Interest Rate



The graph above is repeated below with the legend, matching the data series to the models and historic rate.

Figure 13: Three Model Simulations and Historic Interest Rate with Legend



The graphs above modeled interest rates during the year 2002, which was the low volatility time period used earlier in the paper. The reader can understand the accuracy of these models from the exercise above.

The Vasicek and Cox-Ingersoll-Ross models are only a few of the many interest rate models that have been developed. Each model has different strengths and weaknesses and may work better in different time periods. Most incorporate many of the same elements of the Vasicek and Cox-Ingersoll-Ross models such as a stochastic process (a random element), mean reversion, and a non-constant variance (in the Cox-Ingersoll-Ross model only). These models are a solid building ground for future models that may be able to improve upon some of the shortcomings from these one-factor models, but these models are still be used in practice today.

Appendix A: A Proof for Cox-Ingersoll-Ross Parameter Estimation using the Maximum Likelihood Estimate

Want to find parameters c, h that will minimize

$$\sum_{i=1}^{n} u_i^2 \quad \text{where } u_i = y_i - h - mx_i \text{ and } u_i \sim N(0, \sigma\sqrt{x_i})$$

To do this the maximum likelihood function, L, is needed:

$$L = \prod_{i=1}^{n} f(u_i) \quad \text{where } f(u_i) = \frac{1}{\sigma\sqrt{2\pi x_i}} e^{-\frac{u_i^2}{2\sigma^2 x_i}}$$

$$\ln L = \sum_{i=1}^{n} \ln f(u_i)$$

$$= \sum_{i=1}^{n} \ln\left[\frac{1}{\sigma\sqrt{2\pi x_i}} e^{-\frac{u_i^2}{2\sigma^2 x_i}}\right]$$

$$= \sum_{i=1}^{n} \ln\left[\frac{1}{\sigma\sqrt{2\pi x_i}}\right] + \sum_{i=1}^{n} \ln\left[e^{-\frac{u_i^2}{2\sigma^2 x_i}}\right]$$

$$L = \sum_{i=1}^{n} -\ln[\sigma\sqrt{2\pi x_i}] + \sum_{i=1}^{n} -\frac{u_i^2}{2\sigma^2 x_i}$$

$$L = \sum_{i=1}^{n} -\ln[\sigma\sqrt{2\pi x_i}] + \sum_{i=1}^{n} -\frac{(y_i - h - mx_i)^2}{2\sigma^2 x_i}$$

To minimize this equation, take the partial derivatives and set equal to zero:

$$\frac{\partial L}{\partial h} = 0 = \frac{\partial}{\partial h} \sum_{i=1}^{n} -\frac{(y_i - h - mx_i)^2}{x_i}$$

$$0 = \sum_{i=1}^{n} \frac{y_i}{x_i} - h\sum_{i=1}^{n} \frac{1}{x_i} - mn$$

$$h = \frac{\sum_{i=1}^{n} \frac{y_i}{x_i} - mn}{\sum_{i=1}^{n} \frac{1}{x_i}}$$

$$\frac{\partial L}{\partial m} = 0 = -\frac{1}{2}\sum_{i=1}^{n} \frac{2(y_i - h - mx_i)(-x_i)}{x_i \sigma^2}$$

$$0 = \sum_{i=1}^{n} (y_i - h - mx_i)$$

$$0 = \sum_{i=1}^{n} y_i - hn - m\sum_{i=1}^{n} x_i$$

Substitute h into the second equation to solve for m:

$$m\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i - n\left(\frac{\sum_{i=1}^{n} \frac{y_i}{x_i} - mn}{\sum_{i=1}^{n} \frac{1}{x_i}}\right)$$

$$m\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i - \left(\frac{n\sum_{i=1}^{n} \frac{y_i}{x_i}}{\sum_{i=1}^{n} \frac{1}{x_i}}\right) + \left(\frac{n^2 m}{\sum_{i=1}^{n} \frac{1}{x_i}}\right)$$

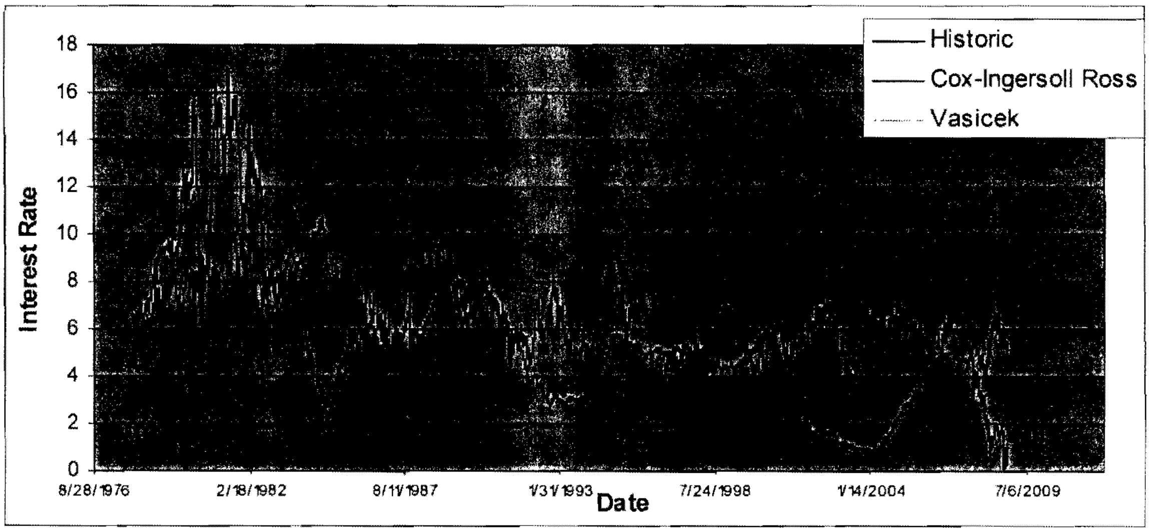
$$m\left(\sum_{i=1}^{n} x_i - \frac{n^2}{\sum_{i=1}^{n} \frac{1}{x_i}}\right) = \sum_{i=1}^{n} y_i - \frac{n\sum_{i=1}^{n} \frac{y_i}{x_i}}{\sum_{i=1}^{n} \frac{1}{x_i}}$$

$$m = \frac{\sum_{i=1}^{n} y_i - \frac{n\sum_{i=1}^{n} \frac{y_i}{x_i}}{\sum_{i=1}^{n} \frac{1}{x_i}}}{\sum_{i=1}^{n} x_i - \frac{n^2}{\sum_{i=1}^{n} \frac{1}{x_i}}}$$

Appendix B: Comparison of Historic Rates versus Vasicek and Cox-Ingersoll-Ross Simulated Rates

Note: In the following models best versions of the models were chosen to compare the Vasicek and Cox-Ingersoll-Ross model in various time spans. Version two of the Vasicek model was used and version four of the Cox-Ingersoll-Ross model was used. While this paper did not determine whether version three or four of the Cox-Ingersoll-Ross model can better represent the historic rates, version four was arbitrarily chosen because use of both would crowd the graphs.

Thirty Year Time Period



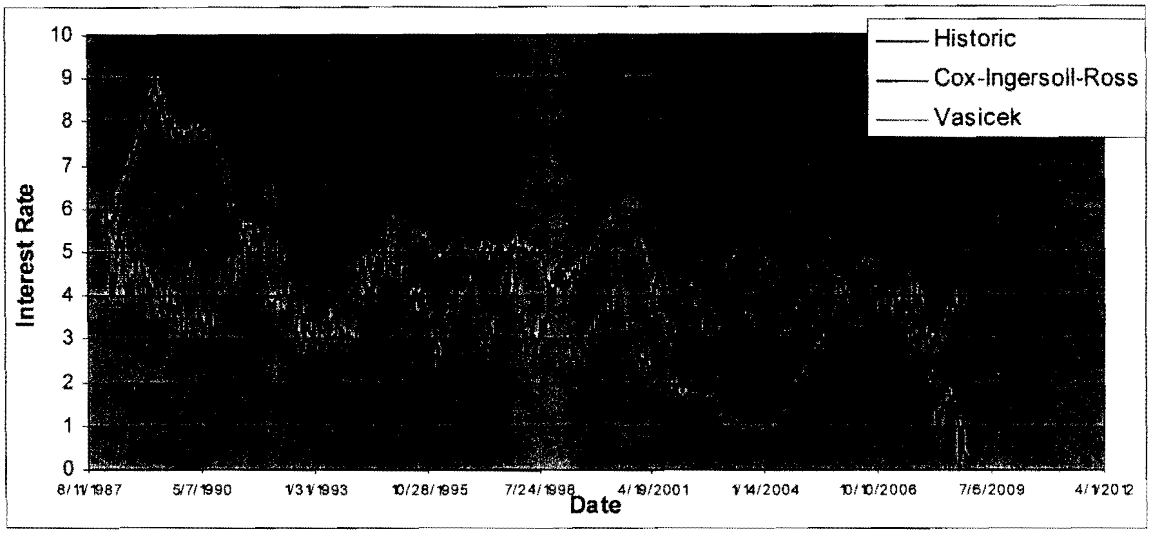
Parameters for r_k

	Vasicek	CIR
a	.1115	.0575
b	4.0188	5.6263
σ	0.0004	0.0035

Statistical Results

	Historic	Vasicek	CIR
mean	5.8434	6.319	5.8479
standard deviation	3.1801	2.0666	3.1777

Twenty Year Time Period



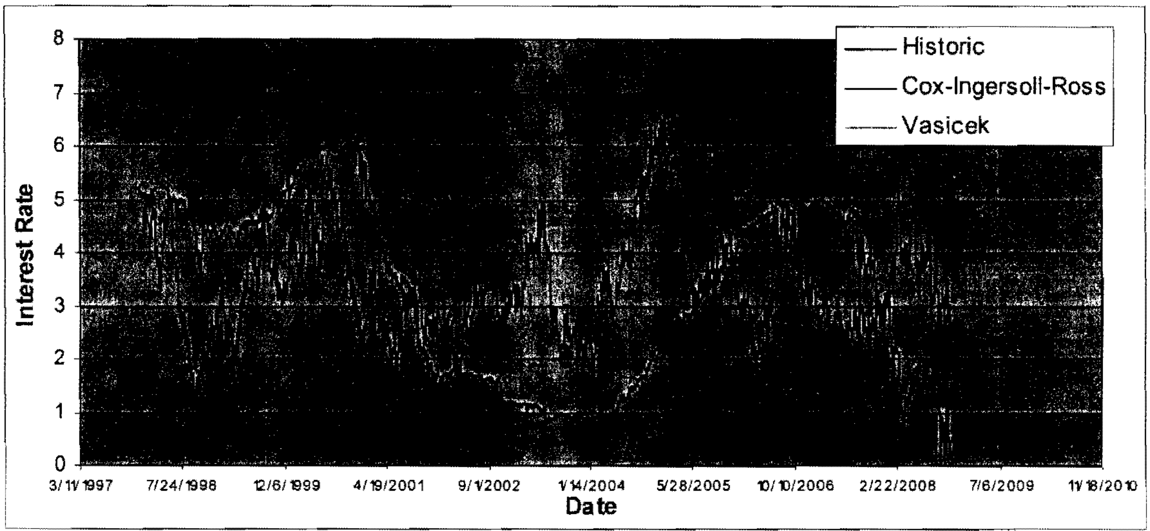
Parameters for r_k

	Vasicek	CIR
a	0.0001	0.0063
b	-3.5660	4.1391
σ	0.0605	0.0594

Statistical Results

	Historic	Vasicek	CIR
mean	4.3082	6.0892	3.8695
standard deviation	1.9551	0.8536	0.7754

Ten Year Time Period



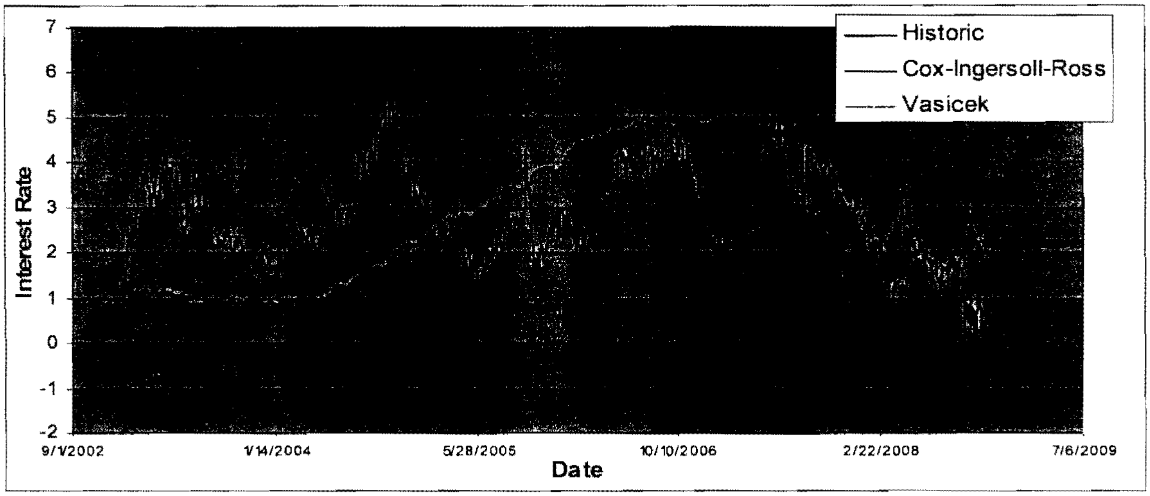
Parameters for r_k

	Vasicek	CIR
a	0.0005	0.0106
b	-0.2923	3.1475
σ	0.0702	0.0799

Statistical Results

	Historic	Vasicek	CIR
mean	3.3166	3.4008	3.4251
standard deviation	1.6831	1.8272	0.9718

Five Year Time Period



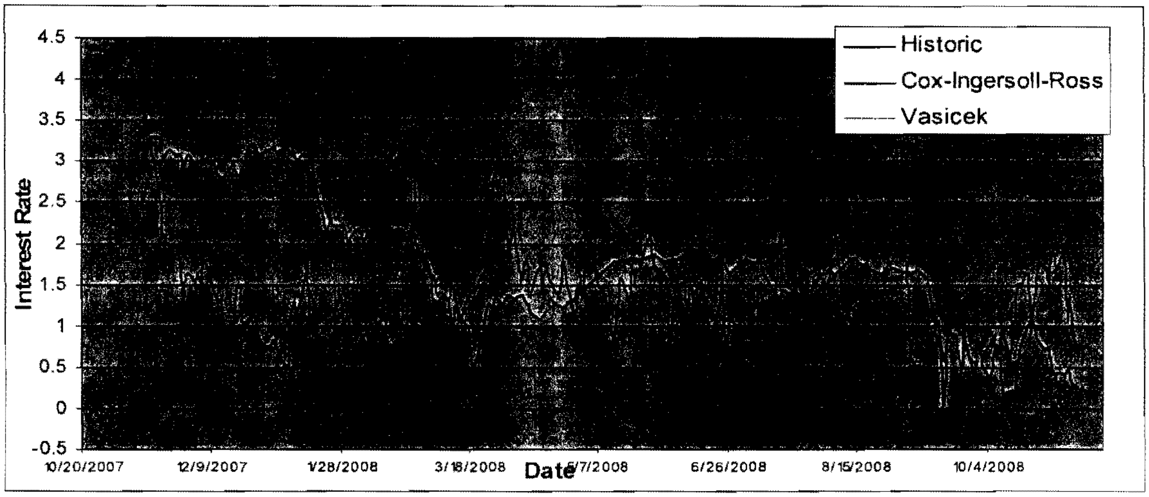
Parameters for r_k

	Vasicek	CIR
a	0.0008	0.0140
b	1.9799	2.8782
σ	0.0803	0.1051

Statistical Results

	Historic	Vasicek	CIR
mean	2.7065	0.6413	2.8019
standard deviation	1.5509	0.8830	0.9560

One Year Time Period



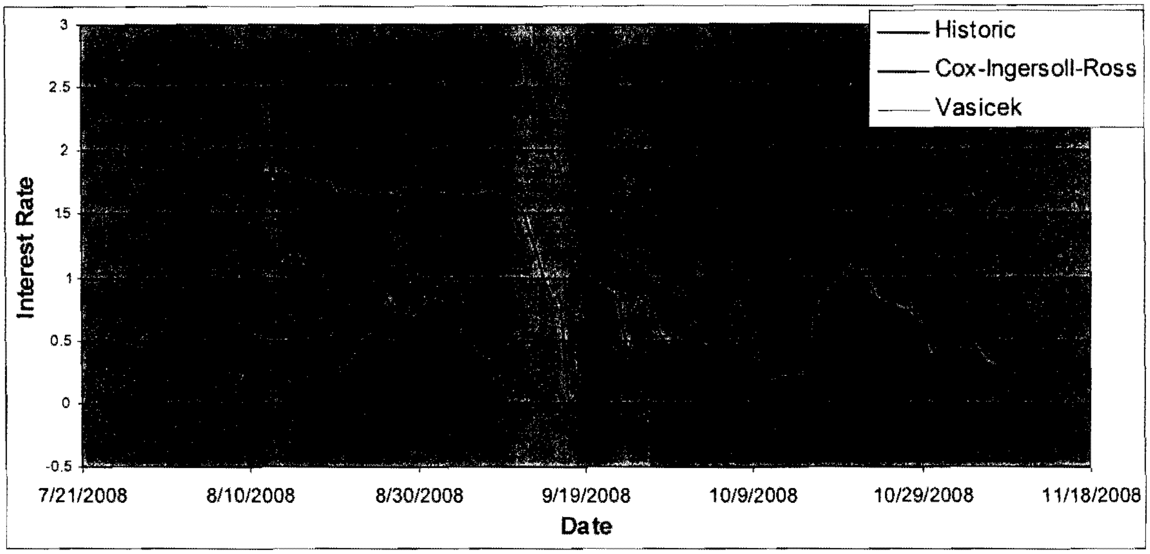
Parameters for r_k

	Vasicek	CIR
a	0.0187	0.0926
b	1.0787	1.5932
σ	0.1425	0.2425

Statistical Results

	Historic	Vasicek	CIR
mean	1.7252	1.5759	1.3551
standard deviation	0.7716	1.0532	0.4535

Three Month Time Period



Parameters for r_k

	Vasicek	CIR
a	0.0787	0.2264
b	0.7144	0.8788
σ	0.2236	0.4725

Statistical Results

	Historic	Vasicek	CIR
mean	1.0016	1.2157	0.8035
standard deviation	0.5807	0.4492	0.5537

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